Math 213, Fall 2025, Oct. 29, 2025
Exam 2
Read Carefully, Present work in an organized way and show all work.
Write Neatly, Sketch Accurately

1. (8 points) Find the limit or show that the limit does not exist
   a. $\lim_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y}$

      Solution: Plugging in gives 0/0; Factor to get

      $$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(1,1)} x+y = 1+1 = 2$$

   b. $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$

      Solution: Here coming to (0,0) from different directions results in different limits.
      Along $y = mx$ we have $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \dfrac{2xmx}{x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \dfrac{2m}{1+m^2}$
      which has different values for different values of *m*.

2. (8 points) If $xy + z^3x - 2yz = 0$ defines *z* as a function of *x* and *y*, find $\dfrac{\partial z}{\partial x}$ at the point (2, 8, 2).

   Solution: Taking the derivative with respect to *x* gives

   $$y + 3z^2\frac{\partial z}{\partial x}x + z^3 - 2y\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-y-z^3}{3z^2x-2y}$$

   $$\Rightarrow \left.\frac{\partial z}{\partial x}\right|_{(2,8,2)} = \frac{-8-2^3}{3\cdot 2^2\cdot 2 - 2\cdot 8} = \frac{-16}{8} = -2$$

   Or using $\dfrac{\partial z}{\partial x} = \dfrac{-F_x}{F_z} = \dfrac{-\left(y+z^3\right)}{3z^2x-2y}$ where $F(x,y,z) = xy + z^3x - 2yz$

3. (12 points) If $xy + z^3x - 2yz = 0$ defines a surface, find the equation of the tangent plane and the normal line (parametric equations) to the surface at the point (2, 8, 2).

   Solution: For the equation of the tangent plane, we first find

   $$\nabla F = \left(y+z^3\right)\mathbf{i} + \left(x-2z\right)\mathbf{j} + \left(3z^2x-2y\right)\mathbf{k}$$

   $$\Rightarrow \left.\nabla F\right|_{(2,8,2)} = (16)\mathbf{i} + (-2)\mathbf{j} + (8)\mathbf{k}$$

   Then the normal line is: $x = 2+16t; y = 8-2t; z = 2+8t; -\infty < t < \infty$

   While the tangent plane is $16(x-2) - 2(y-8) + 8(z-2) = 0$

4. (14 points) Find the directional derivative of $f(x,y,z)=xe^y+z^2$ at $(2,\ln 2,1/2)$ in the direction of $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$. Find the maximum rate of change of f at $(2,\ln 2,1/2)$ and the direction in which it occurs.

Solution: We need to find ∇f at $(2,\ln 2,1/2)$ so

$\nabla f=\left(e^y\right)\mathbf{i}+\left(xe^y\right)\mathbf{j}+(2z)\mathbf{k}\Rightarrow \nabla f\big|_{(2,\ln 2,1/2)}=(2)\mathbf{i}+(4)\mathbf{j}+(1)\mathbf{k}$

Direction of $2\mathbf{i}-3\mathbf{j}+6\mathbf{k}$ is $(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})/\sqrt{4+9+36}=(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})/7$

So directional derivative is $(2\mathbf{i}+4\mathbf{j}+\mathbf{k})\bullet(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})/7=(4-12+6)/7=-2/7$

The maximum rate of change is $\left|\nabla f\right|_{(2,8,2)}=\left|\langle 2,4,1\rangle\right|=\sqrt{2^2+4^2+1^2}=\sqrt{21}$

And its direction is $\langle 2,4,1\rangle/\sqrt{21}=\dfrac{2}{\sqrt{21}}\mathbf{i}+\dfrac{4}{\sqrt{21}}\mathbf{j}+\dfrac{1}{\sqrt{21}}\mathbf{k}$

5. (20 points) Find the critical points of $f(x,y)=6x-8xy+2y+1$ and classify them. Then find the absolute maximum and minimum of f on the closed triangular plate bounded by the lines $x=0; y=0$ and $x+y=2$.

Solution: Set $f_x=0$ and $f_y=0\Rightarrow 6-8y=0$ and $-8x+2=0\Rightarrow x=1/4; y=3/4$. To classify this point use the discriminant $f_{xx}=0; f_{yy}=0; f_{xy}=-8\Rightarrow f_{xx}\cdot f_{yy}-f_{xy}{}^2=-64<0$

So at (1/4,3/4) there is a saddle point. Here f=5/2.

For absolute max and min we need to check the three sides of the triangle:

On $x=0;\ 0\le y\le 2: f=2y+1; \min=1$ at $(0,0)$ max=3 at $(0,2)$ since f linear so max/min at endpoints

On $y=0;\ 0\le x\le 2: f=6x+1; \min=1$ at $(0,0)$ max=13 at $(2,0)$ since f linear so max/min at endpoints

On $x+y=2\Rightarrow y=2-x; 0\le x\le 2; f=6x-8x(2-x)+2(2-x)+1=8x^2-12x+5$

Then $f'=16x-12; f'=0$ at $x=3/4; y=5/4$ Here f= ½. This is a local minimum by second derivative test for one variable.

At the endpoints (0,2) f=3 and at (2,0) f=13.

Conclusion: Absolute max is 13 at (2,0); Absolute min is ½ at (3/4, 5,4)

6. (12 points) Use Lagrange Multipliers to find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Solution: Minimize $f=x^2+y^2+z^2$ subject to $g=x+y+z=9$.

Require

$\nabla f=\lambda\nabla g\Rightarrow 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=\lambda\mathbf{i}+\lambda\mathbf{j}+\lambda\mathbf{k}$

$\Rightarrow \lambda=2x=2y=2z\Rightarrow x=y=z$

$x+y+z=9\Rightarrow 3x=9\Rightarrow x=y=z=3$

7. (12 points) Find the Taylor series *quadratic* expansion for $f(x,y)=e^x\cos y$ around (0, 0) using the methods taught in this course. Use your approximation to approximate $e^{0.2}\cos 0.1$

Solution:

$$f(x,y)\approx f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left(x^2 f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2 f_{yy}(0,0)\right)$$

where

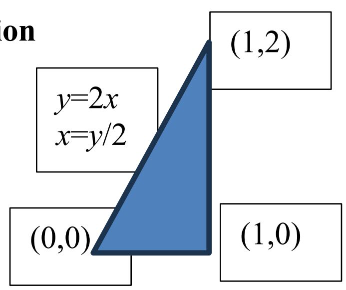
$$f_x=e^x\cos y;\ f_y=-e^x\sin y;\ f_{xx}=e^x\cos y;\ f_{xy}=-e^x\sin y;\ f_{yy}=-e^x\cos y;$$

$$\Rightarrow f(0,0)=1;\ f_x(0,0)=1;\ f_y(0,0)=0;\ f_{xx}(0,0)=1;\ f_{xy}(0,0)=0;\ f_{yy}(0,0)=-1;$$

$$\Rightarrow f(x,y)\approx 1+x+\frac{x^2-y^2}{2!}\Rightarrow f(0.2,0.1)\approx 1+0.2+\frac{0.04-0.01}{2!}=1.215$$

8. (14 points) Sketch the region of integration for $\int_0^2\int_{y/2}^1\cos\left(x^2\right)dxdy$. Then reverse the order of integration and evaluate the integral.

Solution



Reversing the order of integration gives

$$\int_0^1\int_0^{2x}\cos\left(x^2\right)dydx=\int_0^1 \ y\cos\left(x^2\right)\Big|_0^{2x}\ dx=\int_0^1 2x\cos\left(x^2\right)\Big|dx=\sin\left(x^2\right)\Big|_0^1=\sin 1$$